

Wigner's Last Papers on Spacetime Symmetries ¹

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Abstract

Wigner's 1939 paper on representations of the inhomogeneous Lorentz group is one of the most fundamental papers in physics. Wigner maintained his passion for this subject throughout his life. In this report, I will review the papers which he published with me on this subject. These papers deal with the question of unifying the internal space-time symmetries of massive and massless particles.

1 Introduction

I met Eugene Wigner while I was a graduate student at Princeton University from 1958 to 1961. I stayed there for one more year as a post-doctoral fellow before joining the faculty of the University of Maryland in 1962. My advisor was Sam Treiman, and I wrote my PhD thesis on dispersion relations. However, my extra-curricular activity was on Wigner's papers, particularly on his 1939 paper on representations of the Poincaré group [1]. It is not uncommon for one's extra-curricular activity to become his/her life-time job. Indeed, by 1985, I had completed the manuscript for the book entitled *Theory and Applications of the Poincaré Group* [2] with Marilyn Noz who has been my closest colleague since 1970.

After writing this book, I approached Wigner again and asked him whether I could start working on edited volumes of all the papers he had written, but he had a better idea. Wigner told me that he was interested in writing new papers and that he had been looking for a younger person who could collaborate with him. This was how I was able to publish seven papers with him. Today, I would like to talk about two of those papers. They constitute a re-interpretation of Wigner's original paper on the Poincaré group.

Why is this paper so important? Where does it stand in the history of physics? From the principles of special relativity, Einstein derived the relation $E = mc^2$ in 1905. This formula unifies the momentum-energy relations for both massive and massless particles, which are $E = p^2/2m$ and $E = cp$ respectively. In his 1939

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Figure 1: Eugene Wigner and Albert Einstein. Portrait by Bulent Atalay (1978).

paper [1], Wigner observed that relativistic particles have their internal space-time degrees of freedom. For example, the spin of a particle at rest is a manifestation of the three-dimensional rotational symmetry. Wigner in his paper formulated space-time symmetries of relativistic particles in terms of the little groups of the Poincaré group. In this review talk, I would like to emphasize that Wigner’s little group is a Lorentz-covariant entity and unifies the internal space-time symmetries of both massive and massless particles, just as Einstein’s $E = mc^2$ does for the energy-momentum relation.

On the other hand, Wigner did not reach this conclusion in 1939, and the above statement is based on many subsequent papers published on this subject during the period 1939-1990. In fact, his 1939 paper has a stormy history. This paper had been rejected by one of the prestigious mathematics journals before John von Neumann, then the editor of the Annals of Mathematics, invited Wigner to submit it to his journal. It is not uncommon even these days to hear the comment that the paper does not have anything to do with physics. Today, I would like to clarify this issue.

In Sec. 2, I give a brief review of the subject and explain why Wigner’s paper is essential in understanding modern physics. In order to give a more transparent interpretation of his paper, I give a geometrical interpretation of his work based on the paper which Wigner published with me in 1987 and 1990 [3, 4]. The purpose of these two papers was to translate all the earlier works on this subject into a geometrical language. The main conclusion of these papers is that the $E(2)$ -like little group does not share the same geometry as the $E(2)$ group whose geometry is quite transparent to us. The geometry of the little group is that is the cylindrical group dealing with the surface of a circular cylinder [5]. The cylindrical axis is parallel to the momentum.

Also shown in these two papers is that the $O(3)$ -like little group, which can be described in terms of a sphere in the rest frame, becomes continuously deformed into the symmetry group describing a point moving on the cylindrical surface as the mo-

mentum/mass ratio becomes large. For the case of electromagnetic four-potential satisfying the Lorentz condition, the rotation around the axis corresponds to helicity, while the translation along the direction of the axis corresponds to a gauge transformation.

In Sec. 3, we discuss the three-dimensional rotation group and its contractions to the cylindrical and the two-dimensional Euclidean groups. It is shown that both of these contractions can be combined into a single four-by-four representation. In Sec. 4, the generators of the little groups are discussed in the light-cone coordinate system. It is shown that these generators are identical with the combined geometry of the cylindrical group and the Euclidean group discussed in Sec. 3. The geometry of Sec. 3 therefore gives a comprehensive description of the little groups for massive and massless particles.

2 Historical Review of Wigner’s Little Groups

In 1939, Wigner observed that internal space-time symmetries of relativistic particles are dictated by their respective little groups [1]. The little group is the maximal subgroup of the Lorentz group which leaves the four-momentum of the particle invariant. He showed that the little groups for massive and massless particles are isomorphic to the three-dimensional rotation group and the two-dimensional Euclidean group respectively. Wigner’s 1939 paper indeed gives a covariant picture massive particles with spins, and connects the helicity of massless particle with the rotational degree of freedom in the group $E(2)$. This paper also gives many homework problems for us to solve.

- First, like the three-dimensional rotation group, $E(2)$ is a three-parameter group. It contains two translational degrees of freedom in addition to the rotation. What physics is associated with the translational-like degrees of freedom for the case of the $E(2)$ -like little group?
- Second, as is shown by Inonu and Wigner [6], the rotation group $O(3)$ can be contracted to $E(2)$. Does this mean that the $O(3)$ -like little group can become the $E(2)$ -like little group in a certain limit?
- Third, it is possible to interpret the Dirac equation in terms of Wigner’s representation theory [7]. Then, why is it not possible to find a place for Maxwell’s equations in the same theory?
- Fourth, the proton was found to have a finite space-time extension in 1955 [8], and the quark model has been established in 1964 [9]. The concept of relativistic extended particles has now been firmly established. Is it then possible to

construct a representation of the Poincaré group for particles with space-time extensions?

The list could be endless, but let us concentrate on the above four questions. As for the first question, it has been shown by various authors that the translation-like degrees of freedom in the $E(2)$ -like little group is the gauge degree of freedom for massless particles [10]. As for the second question, it is not difficult to guess that the $O(3)$ -like little group becomes the $E(2)$ -like little group in the limit of large momentum/mass [11]. However, the non-trivial result is that the transverse rotational degrees of freedom become gauge degrees of freedom [12].

Then there comes the third question. Indeed, in 1964 [13], Weinberg found a place for the electromagnetic tensor in Wigner's representation theory. He accomplished this by constructing from the $SL(2, c)$ spinors all the representations of massless fields which are invariant under the translation-like transformations of the $E(2)$ -like little group. Since the translation-like transformations are gauge transformations, and since the electromagnetic tensor is gauge-invariant, Weinberg's construction should contain the electric and magnetic fields, and it indeed does.

Next question is whether it is possible to construct electromagnetic four-potentials. After identifying the translation-like degrees of freedom as gauge degrees of freedom, this becomes a tractable problem. It is indeed possible to construct gauge-dependent four-potentials from the $SL(2, c)$ spinors [14]. Yes, both the field tensor and four-potential now have their proper places in Wigner's representation theory. The Maxwell theory and the Poincaré group are perfectly consistent with each other.

The fourth question is about whether Wigner's little groups are applicable to high-energy particle physics where accelerators produce Lorentz-boosted extended hadrons such as high-energy protons. The question is whether it is possible to construct a representation of the Poincaré group for hadrons which are believed to be bound states of quarks [2, 15]. This representation should describe Lorentz-boosted hadrons. Next question then is whether those boosted hadrons give a description of Feynman's parton picture [16] in the limit of large momentum/mass. These issues have also been discussed in the literature [2, 17].

The application of the Poincaré group is not limited to relativistic theories of particles. This group plays many important roles in classical mechanics, the theory of superconductivity, as well as in quantum optics. This new trend makes it more urgent to understand correctly Wigner's papers on the Lorentz group. The following sections are based on Wigner's last papers on this subject [3, 4] where his 1939 paper was translated into a geometrical language.

3 Three-dimensional Geometry of the Little Groups

The little groups for massive and massless particles are isomorphic to $O(3)$ and $E(2)$ respectively. It is not difficult to construct the $O(3)$ -like geometry of the little group for a massive particle at rest [1]. The generators L_i of the rotation group satisfy the commutation relations:

$$[L_i, L_j] = i\epsilon_{ijk}L_k. \quad (1)$$

Transformations applicable to the coordinate variables x, y , and z are generated by

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (2)$$

The Euclidean group $E(2)$ is generated by L_3, P_1 and P_2 , with

$$P_1 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}, \quad (3)$$

and they satisfy the commutation relations:

$$[P_1, P_2] = 0, \quad [L_3, P_1] = iP_2, \quad [L_3, P_2] = -iP_1. \quad (4)$$

The generator L_3 is given in Eq.(2). When applied to the vector space $(x, y, 1)$, P_1 and P_2 generate translations on in the xy plane. The geometry of $E(2)$ is also quite familiar to us.

Let us transpose the above algebra. Then P_1 and P_2 become Q_1 and Q_2 , where

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix}, \quad (5)$$

respectively. Together with L_3 , these generators satisfy the same set of commutation relations as that for L_3, P_1 , and P_2 given in Eq.(4)

$$[Q_1, Q_2] = 0, \quad [L_3, Q_1] = iQ_2, \quad [L_3, Q_2] = -iQ_1. \quad (6)$$

These matrices generate transformations of a point on a circular cylinder. Rotations around the cylindrical axis are generated by L_3 . The Q_1 and Q_2 matrices generate the transformation:

$$\exp(-i\xi Q_1 - i\eta Q_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \xi & \eta & 1 \end{pmatrix}. \quad (7)$$

When applied to the space (x, y, z) , this matrix changes the value of z while leaving the x and y variables invariant [3]. This corresponds to a translation along the

cylindrical axis. The J_3 matrix generates rotations around the axis. We shall call the group generated by J_3, Q_1 and Q_2 the *cylindrical group*.

We can achieve the contractions to the Euclidean and cylindrical groups by taking the large-radius limits of

$$P_1 = \frac{1}{R} B^{-1} L_2 B, \quad P_2 = -\frac{1}{R} B^{-1} L_1 B, \quad (8)$$

and

$$Q_1 = -\frac{1}{R} B L_2 B^{-1}, \quad Q_2 = \frac{1}{R} B L_1 B^{-1}, \quad (9)$$

where

$$B(R) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{pmatrix}.$$

The vector spaces to which the above generators are applicable are $(x, y, z/R)$ and (x, y, Rz) for the Euclidean and cylindrical groups respectively. They can be regarded as the north-pole and equatorial-belt approximations of the spherical surface respectively.

Since $P_1(P_2)$ commutes with $Q_2(Q_1)$, we can consider the following combination of generators.

$$F_1 = P_1 + Q_1, \quad F_2 = P_2 + Q_2. \quad (10)$$

Then these operators also satisfy the commutation relations:

$$[F_1, F_2] = 0, \quad [L_3, F_1] = iF_2, \quad [L_3, F_2] = -iF_1. \quad (11)$$

However, we cannot make this addition using the three-by-three matrices for P_i and Q_i to construct three-by-three matrices for F_1 and F_2 , because the vector spaces are different for the P_i and Q_i representations. We can accommodate this difference by creating two different z coordinates, one with a contracted z and the other with an expanded z , namely $(x, y, Rz, z/R)$. Then the generators become four-by-four matrices, and F_1 and F_2 take the form

$$F_1 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (12)$$

The rotation generator L_3 is also a four-by-four matrix:

$$L_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (13)$$

These four-by-four matrices satisfy the $E(2)$ -like commutation relations of Eq.(11).

Next, let us consider the transformation matrix generated by the above matrices. It is easy to visualize the transformations generated by P_i and Q_i . It would be easy to visualize the transformation generated by F_1 and F_2 , if P_i commuted with Q_i . However, P_i and Q_i do not commute with each other, and the transformation matrix takes a somewhat complicated form:

$$\exp \{-i(\xi F_1 + \eta F_2)\} = \begin{pmatrix} 1 & 0 & 0 & \xi \\ 0 & 1 & 0 & \eta \\ \xi & \eta & 1 & (\xi^2 + \eta^2)/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (14)$$

4 Little Groups in the Light-cone Coordinate System

Let us now study the group of Lorentz transformations using the light-cone coordinate system. If the space-time coordinate is specified by (x, y, z, t) , then the light-cone coordinate variables are (x, y, u, v) for a particle moving along the z direction, where

$$u = (z + t)/\sqrt{2}, \quad v = (z - t)/\sqrt{2}. \quad (15)$$

The transformation from the conventional space-time coordinate to the above system is achieved through a similarity transformation.

It is straight-forward to write the rotation generators J_i and boost generators K_i in this light-cone coordinate system [4]. If a massive particle is at rest, its little group is generated by J_1, J_2 and J_3 . For a massless particle moving along the z direction, the little group is generated by N_1, N_2 and J_3 , where

$$N_1 = K_1 - J_2, \quad N_2 = K_2 + J_1, \quad (16)$$

which can be written in the matrix form as

$$N_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad N_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (17)$$

and J_3 takes the form of the four-by-four matrix given in Eq.(13)

These matrices satisfy the commutation relations:

$$[J_3, N_1] = iN_2, \quad [J_3, N_2] = -iN_1, \quad [N_1, N_2] = 0. \quad (18)$$

Let us go back to F_1 and F_2 of Eq.(12). Indeed, they are proportional to N_1 and N_2 respectively. Since F_1 and F_2 are somewhat simpler than N_1 and N_2 , and since the

commutation relations of Eq.(18) are invariant under multiplication of N_1 and N_2 by constant factors, we shall hereafter use F_1 and F_2 for N_1 and N_2 .

In the light-cone coordinate system, the boost matrix takes the form

$$B(R) = \exp(-i\rho K_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & 1/R \end{pmatrix}, \quad (19)$$

with $\rho = \ln(R)$, and $R = \sqrt{(1+\beta)/(1-\beta)}$, where β is the velocity parameter of the particle. The boost is along the z direction. Under this transformation, x and y coordinates are invariant, and the light-cone variables u and v are transformed as

$$u' = Ru, \quad v' = v/R. \quad (20)$$

If we boost J_2 and J_1 and multiply them by $\sqrt{2}/R$, as

$$W_1(R) = -\frac{\sqrt{2}}{R} B J_2 B^{-1} = \begin{pmatrix} 0 & 0 & -i/R^2 & i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ i/R^2 & 0 & 0 & 0 \end{pmatrix},$$

$$W_2(R) = \frac{\sqrt{2}}{R} B J_1 B^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i/R^2 & i \\ 0 & i & 0 & 0 \\ 0 & i/R^2 & 0 & 0 \end{pmatrix}, \quad (21)$$

then $W_1(R)$ and $W_2(R)$ become F_1 and F_2 of Eq.(12) respectively in the large- R limit.

The most general form of the transformation matrix is

$$D(\xi, \eta, \alpha) = D(\xi, \eta, 0)D(0, 0, \alpha), \quad (22)$$

with

$$D(\xi, \eta, 0) = \exp\{-i(\xi F_1 + \eta F_2)\}, \quad D(0, 0, \alpha) = \exp(-i\alpha J_3). \quad (23)$$

The matrix $D(0, 0, \alpha)$ represents a rotation around the z axis. In the light-cone coordinate system, $D(\xi, \eta, 0)$ takes the form of Eq.(14). It is then possible to decompose it into

$$D(\xi, \eta, 0) = C(\xi, \eta)E(\xi, \eta)S(\xi, \eta), \quad (24)$$

where

$$S(\xi, \eta) = I + \frac{1}{2} [C(\xi, \eta), E(\xi, \eta)] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & (\xi^2 + \eta^2)/2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{aligned}
E(\xi, \eta) &= \exp(-i\xi P_1 - i\eta P_2) = \begin{pmatrix} 1 & 0 & 0 & \xi \\ 0 & 1 & 0 & \eta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
C(\xi, \eta) &= \exp(-i\xi Q_1 - i\eta Q_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \xi & \eta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned} \tag{25}$$

Let us consider the application of the above transformation matrix to an electromagnetic four-potential of the form

$$A^\mu(x) = A^\mu e^{i(kz - \omega t)}, \tag{26}$$

with

$$A^\mu = (A_1, A_2, A_u, A_v), \tag{27}$$

where $A_u = (A_3 + A_0)/\sqrt{2}$, and $A_v = (A_3 - A_0)/\sqrt{2}$. If we impose the Lorentz condition, the above four-vector becomes

$$A^\mu = (A_1, A_2, A_u, 0), \tag{28}$$

The matrix $S(\xi, \eta)$ leaves the above four-vector invariant. The same is true for the $E(\xi, \eta)$ matrix. Both $E(\xi, \eta)$ and $S(\xi, \eta)$ become identity matrices when applied to four-vectors with vanishing fourth component. Thus only the $C(\xi, \eta)$ matrix performs non-trivial operations. As in the case of Eq.(7), it performs transformations parallel to the cylindrical axis, which in this case is the direction of the photon momentum. It leaves the transverse components of the four vector invariant, but changes the longitudinal and time-like components at the same rate. This is a gauge transformation.

It is remarkable that the algebra of Lorentz transformations given in this section can be explained in terms of the geometry of deformed spheres developed in Sec. 3.

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